Generalized Erlang-B model in case of state-dependent arrival and departure processes
Seferin T. Mirtchev

In this article, a generalization of the classic full availability multi-server loss system with generalized Poisson arrival process and generalized Bernoulli departure process is proposed. The generalized full availability loss system has a nonlinear dependence of the arrival and departure intensities from the system states, which allow defining different arrival and departure flows with two parameters – mean value and variance. For the generalized system, formulas are defined and the state probabilities and the time congestion probabilities are calculated and shown graphically. Investigated teletraffic system is marked by Kendall notation as - Mg/Mg/n/0/S and it is described by a birth and death process. The proposed approach allows with one model to study full availability multi-server loss systems with smooth, regular and peaked distribution of the system states. The numerical results and experience show that the proposed generalized Erlang-B model has interesting features and it is useful for analysis of teletraffic systems.

1. Introduction

One of the most widespread teletraffic distributions used in the modeling and design of telecommunications networks is exponential. Usually it is assumed that the incoming stream of calls is Poisson, and that the service time is exponentially distributed. In practice, much of modern traffic sources in the networks generate peaked traffic that has long term dependences. As a result, the models and methods, giving an account of traffic flows peakedness, became very important for the study and planning of broadband telecommunications networks [1].

Variety of models and methods for description of random processes in advanced broadband networks is developed and analyzed, as it is not possible to propose a single model describing traffic characteristics of all networks. The development of better and more appropriate models is a key requirement in the planning of such networks. In [2] is proposed a model of a full availability loss system with retrials based mainly on the analysis of traffic sources. For analysis of broadband networks with peaked traffic flows can be used Pareto distribution, which reflects the long history dependencies.
The Bernoulli-Poisson-Pascal (BPP) method is used to evaluate the congestion probabilities of tel- etraffic systems associated in case of smooth, regular and peaked traffic [3]. By this method the smooth, regular and peaked traffic is presented through three separate models and there are limits to any smooth traffic. The traffic model of Markov Modulated Pois- son Process (MMPP), which accurately approximates internet traffic, is presented in [4].

To evaluate the influence of the traffic inflows peakedness on the quality of service in broadband networks it is suggested to use a nonlinear dependence of the arrival and departure intensity from the system states. The aim of this article is to generalize the classic full availability loss system by generalized Poisson arrival process and generalized Bernoulli departure process through a nonlinear dependence of the arrival and departure intensities from the system states.

2. Full availability multi-server loss system

Various models of full availability loss systems with identical servers were tested and analyzed for decades [5]. The calculation of the congestion probability and its minimizing is a difficult task. The most famous classical model of loss system M/M/n/0 has been studied for the first time in Erlang (1917). He obtained the following formula (Erlang-B formula) for the loss probability

\[ B = \frac{A^n}{n!} \sum_{i=0}^{n} \frac{A^i}{i!}, \]

where \( A = \lambda \tau \) is offered traffic, \( n \) is the number of the servers.

\( \lambda / \tau \) and \( \tau \) are respectively the average length of time between the moments of arrivals and the average service time, both of which are exponentially distributed. This formula is known as the first or Erlang-B formula.

Palm (1943) is investigated the system Gi/M/n/0 with general distributing independent interval time. He is analyzed the lost calls flow and calculated the probability of losses in the system as follows

\[ \frac{L}{P_a} = \sum_{i=0}^{n} \frac{n}{i!} c_i, \]

where \( c_i \) are constants calculated in the following way

\[ c_0 = \lambda; \quad c_i = \prod_{i=1}^{n} \frac{1 - \varphi \left( \frac{\mu}{\lambda} \right)}{\varphi \left( \frac{\mu}{\lambda} \right)} \quad 1 \leq i \leq n \]

where \( \varphi \) is the Laplace-Stiltes transformation of the intervals between moments of arrival.

Visual evidence of formula 3 is made by Takacs (1957) by using the method of finite difference equations.

The full availability multi-server loss systems GI/M/m/0 is a model with special attention. It is known as a good model for telephone systems and is the subject of study for many decades. It is investigated intensively multi-server loss systems with repeated attempts and retrials in connection with the analysis of call centers and cellular networks. Recently, it is focus on nonstationary multi-server loss systems and of such systems with heterogeneous servers [6, 7].

In [8] it is studied multi-server loss system GI/M/m/0 at heterogeneous services. The first formula of Erlang in state dependant arrival intensity is summarized in [9]. In [10] it is proposed generalization of the Erlang-B formula by the Polya distribution.

In [11] it is investigated a queueing network of multi-server loss systems by analyzing individual queue with GI/G/s/0 system. It is using approximations to obtain the state probabilities of the system.

Despite the large number of studies in the literature on the loss system GI/G/c/0 simple expression for the state probabilities has not been received.

3. Generalized Erlang-B model description

According to the Kendall notation the studied tel- etraffic systems recorded symbolically as Mg/Mg/n/0/S. It represents full availability multi-server loss system with generalized Poisson arrival process Mg, generalized Bernoulli departure process Mg, number of servers \( n \), waiting position \( \theta \) and number of sources \( S \) \((S>n)\).

This multi-server loss system with generalized arrival and departure processes, which can be smooth, regular and peaked, is a birth and death process with nonlinear dependence of the arrival and departure intensities from the system states. It is possible to use the general solution of the birth and death processes for calculation of the states probabilities of the generalized multi-server loss system.

Let’s look at one full availability loss system with \( n \) number of servers and generalized arrival and departure processes for which \( \lambda \) is the arrival intensity when the system is empty, \( p \) is an parameter – peakedness factor, characterizing the nonlinear dependence of generalized Poisson arrival flow from the system stats, \( \mu \) is the service intensity when only one server is busy and \( q \) is an parameter – peakedness factor, characterizing the nonlinear dependence of generalized Bernoulli departure process from the system stats (fig.1). The investigated multi-server loss system with generalized

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arrival and departure processes is described by following arrival and departure intensities

\[ \lambda_k = \lambda (k + 1)^{1/p} \quad k = 0,1,2,\ldots,n \]

(4)

\[ \mu_k = k \mu k^{-q} \quad k = 1,2,3,\ldots,n \]

Arrival process

\[ \lambda, p, S \]

Arrival

\[ A = \lambda/\mu \]

Departure process

\[ \mu, q \]

Servers

\[ n \]

Fig.1. Generalized multi-server loss system - M(g)/M(g)/n/0/S.

The generalized Poisson arrival processes are said to be peaked, regular or smooth according to whether peakedness factor \( p > 1, p = 1 \) or \( p < 1 \), respectively. The generalized Bernoulli departure processes are said to be peaked, regular or smooth according to whether peakedness factor \( q > 1, q = 1 \) or \( q < 1 \), respectively.

The finite state-transition diagram of the investigated multi-server loss system with generalized arrival and service rates is shown in Fig.2.

The arrival \( \lambda_k \) and service \( \mu_k \) intensities depend on the system states by the peakedness factors \( p \) and \( q \) respectively. This teletraffic system always has stationary behavior.

4. Generalized first Erlang distribution

Applying the presented coefficients for the general solution of the birth and death process and using traffic intensity when the system is empty \( a = \lambda/\mu \) we obtain the steady state probabilities of the generalized first Erlang distribution

\[ p_k = \frac{a^k}{(k!)^{1/p-q+1}} \sum_{i=0}^n \left( \frac{a}{(i!)^{1/p-q+1}} \right) k = 0,1,2,\ldots,n \]

The state probabilities are described by generalized first Erlang distribution, which is generalized truncated Poisson distribution. When the peakedness factors, defined the peakedness of the arrival and service processes, are equal to one \( p = q = 1 \), we get the first discrete Erlang distribution.

After calculating the states probabilities it can easy calculate the mean value and the variance of the number of busy servers by formulas for their definition.

The offered traffic is calculated using the average arrival intensity \( \lambda \) and the mean service intensity \( \mu \)

\[ A = \lambda/\mu \]

(6)

(7)

(8)

The carried traffic is equal to the average number of simultaneous calls in the system

\[ A_c = \sum_{k=0}^n k P_k \]

(9)

The time congestion probability \( B_t \) describes the fraction of time that all \( n \) servers of the loss system are busy

\[ B_t(a,p,q,n) = P_{n} \]

(10)

We assume that traffic flow is generated by a finite number of sources \( S \). Usually in practice, \( S >> n \). We
assume that each source has a server. Then this system is ideal (without losses and delays). The offered traffic is equal to the carried and it is called intended traffic \( A_i \). For the ideal system the states probabilities are determined analogous to the formula (5)

\[
P_k = \sum_{j=0}^{\lfloor k/\mu \rfloor} \frac{a_j \lambda_{j-1}}{j!} \left( \frac{\lambda}{\mu} \right)^j \quad 0 \leq k \leq S,
\]

The intended traffic is the equilibrium number of busy servers

\[
A_i = \sum_{j=1}^{S} \lambda_j P_j, \quad \text{Erl},
\]

The variance of the intended traffic is

\[
V(A_i) = \sum_{j=1}^{S} (j - A_i)^2 \lambda_j P_j, \quad \text{Erl}^2,
\]

The peakedness of the intended traffic is the variance to mean ratio

\[
z = \frac{V(A_i)}{A_i},
\]

It is necessary to define the traffic intensity \( \lambda \) (by \( \lambda \) - arrival intensity when the system is empty and \( \mu \) - service intensity when only one server is busy) and the peakedness factors \( p \) and \( q \) for calculation of the state probabilities by formula (5). Then it is easy to calculate the average arrival and service intensities \( \overline{\lambda} \) (7) and \( \overline{\mu} \) (8) and to determine the offered traffic \( A \) by formula (6). In many cases from the practical point of view it is better to define the offered traffic and then analyze the system. For evaluation of the generalized Erlang-B model it is possible to define the intended traffic \( A_i \) and the peakedness factors \( p \) and \( q \) and then to calculate the traffic intensity \( a \) by the method of successive approximations. To calculate the intended traffic, it is necessary to define a finite number of sources \( S \).

5. Numerical results

To obtain numerical results and to present them in graphical form it is written a computer program on a personal computer. The proposed method for evaluation of the multi-server loss systems has been tested on a computer over a wide range of arguments. Figure 3 shows the state probabilities \( P_i \) of the generalized first Erlang distribution with different values of the equal peakedness factors \( p = q \) when the number of the sources is 200 and the intended traffic is 15 Erl. It is seen that the larger values of the peakedness factors (greater variances) lead to the greater values of the probabilities, which are removed from the mean value, which is equal to 15.

On Figure 4 it is presented the dependence of the time congestion probability \( B_i \) from the intended traffic on one server \( A/n \) when the number of the sources is 200, the number of the servers is 20 and different values of the equal peakedness factors \( p = q \).

The figure shows that when the intended traffic is 10 Erl (intended traffic per server is 0.5 Erl) the time congestion probability increase by more than an order of magnitude by changing the peakedness factors from 0.8 to 1.2 (change the generalized Poisson arrival and Bernoulli departure processes from smooth to peaked).

6. Conclusion

In this article it is made a generalization of the Erlang-B formula. It is proposed a method for analyzing generalized full availability multi-server loss system with generalized Poisson arrival and Bernoulli departure processes through a nonlinear dependence of the arrival and departure intensities from the system states. The proposed method allows with a single model to define peaked, regular or smooth arrival processes and also peaked, regular or smooth departure processes. The presented numerical results and the subsequent experience showed that this method is accurate and useful for analysis of teletraffic systems.

There are differences between presented results and calculated losses by the method of Equivalent Random Theory - ERT and the method by Bernoulli-Poisson-Pascal - BPP. The ERT model uses equivalent traffic as a generator of peaked overflow traffic, which is a part of the Poisson distribution. The BPP model uses linear dependence of the arrival intensity from the system states. An important advantage of the presented results of full availability multi-server loss system with generalized Poisson arrival and Bernoulli departure processes comes from the ability to describe behaviors into more complex real queueing systems. In this case, only with one generalized model is describing the behavior of the real teletraffic systems, which is useful in the design of telecommunications networks and systems.
Fig. 3. State probabilities of the generalized first Erlang distribution for the system Mg/Mg/S/0/S.

Fig. 4. Changing the time congestion probabilities at different values of the peakedness factors for the system Mg/Mg/S/0/S.
REFERENCES


Assoc. Prof. DSc Seferin T. Mirtchev has graduated telecommunications at Technical University of Sofia in 1981. He is with Department of Communication Networks, vice president of the Union of Electronics, Electrical Engineering and Telecommunications (CEEC), member of IEEE and has research interest in teletraffic engineering, switching systems, quality of service.

tel.: +359 2 965 2254 e-mail: stm@tu-sofia.bg

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